LIFTING REPRESENTATIONS OF FINITE REDUCTIVE GROUPS: A CHARACTER RELATION

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Abstract. Given a connected reductive group $\widetilde{G}$ over a finite field k, and a semisimple k-automorphism ε of $\widetilde{G}$ of finite order, let G denote the connected part of the group of ε-fixed points. Then there exists a lifting from packets of representations of $G(k)$ to packets for $\widetilde{G}(k)$. In the case of Deligne-Lusztig representations, we show that this lifting satisfies a character relation analogous to that of Shintani.

0. Introduction

Suppose k is a finite field, $\widetilde{G}$ is a connected reductive k-group, and ε is a semisimple k-automorphism of $\widetilde{G}$ of finite order ℓ. From [6, Theorem 7.5], ε must preserve some pair $(\widetilde{B},\widetilde{T})$ consisting of a Borel subgroup of $\widetilde{B} \subseteq \widetilde{G}$ and a maximal torus $\widetilde{T} \subseteq \widetilde{B}$. Call such a pair a *Borel-torus pair* for $\widetilde{G}$. Let G be the connected part of the group $\widetilde{G}^\varepsilon$ of ε-fixed points of $\widetilde{G}$. Then we have the following from [1, Proposition 3.5].

Proposition 1.

- *G is a connected reductive k-group.*
- *For every ε-invariant Borel-torus pair $(\widetilde{B},\widetilde{T})$ for $\widetilde{G}$, one has a Borel-torus pair $(\widetilde{B}^\varepsilon,(\widetilde{T}^\varepsilon)^\circ)$ for G. Moreover, $(\widetilde{T}^\varepsilon)^\circ = \widetilde{T}\cap G$.*
- *For every Borel-torus pair (B,T) for G, one has an ε-invariant Borel-torus pair $(\widetilde{B},\widetilde{T})$ where $\widetilde{T} = C_{\widetilde{G}}(T)$.*

Note that by choosing T to be defined over k, we can show that $\widetilde{G}$ has an ε-invariant Borel-torus pair $(\widetilde{B},\widetilde{T})$ whose torus $\widetilde{T}$ is defined over k.

Let $\widetilde{G}^*$ and G^* denote the duals of $\widetilde{G}$ and G. For each semisimple element $s \in G$, one obtains a collection $\mathcal{E}_s(G(k))$ of irreducible representations of $G(k)$, and these collections, known as *Lusztig series*, partition the set $\mathcal{E}(G(k))$ of (equivalence classes of) irreducible representations of $G(k)$ [5, §14.1]. Suppose that s is regular, and let $T^* \subseteq G^*$ be the unique maximal k-torus containing s. Then the pair (T^*,s) corresponds to a pair (T,θ), where $T \subseteq G$ is a maximal k-torus, and θ is a character of $T(k)$. This latter pair is uniquely determined up to $G(k)$-conjugacy. The Lusztig series $\mathcal{E}_s(G(k))$ corresponding to s is the set of irreducible components of the Deligne-Lusztig virtual representation whose character is $\mathrm{R}_T^G\,\theta$.

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In an earlier work [1, Corollary 11.3], two of the authors constructed a map from semisimple classes in $G^*(k)$ to semisimple classes in $\widetilde{G}^*(k)$, thus lifting each Lusztig series for $G(k)$ to one for $\widetilde{G}(k)$. The series of representations coming from $\pm\,\mathrm{R}_T^G\,\theta$ lifts to that coming from $\pm\,\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta}$, where $\widetilde{T} = C_{\widetilde{G}}(T)$ and $\widetilde{\theta} = \theta\circ\mathcal{N}$, and $\mathcal{N}\colon \widetilde{T} \longrightarrow T$ is the norm map defined by

$$\mathcal{N}(t) = t\varepsilon(t)\cdots\varepsilon^{\ell-1}(t).$$

(Of course, one could define a similar map $\mathcal{N}$ on any ε-invariant torus in $\widetilde{G}$.) In order to understand better this lifting of representations, one would like to have a relation between the character $\mathrm{R}_T^G\,\theta$ (for θ an arbitrary character of $T(k)$, not necessarily associated to a regular element of $G^*(k)$) and the ε-twisted character $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_\varepsilon$ associated to the ε-invariant character $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta}$. The purpose of the present paper is to prove that such a relation holds at sufficiently regular points.

Theorem. *Suppose $\widetilde{s} \in \widetilde{G}(k)$ belongs to an ε-invariant, maximal k-torus and that $\mathcal{N}(\widetilde{s})$ is regular in $\widetilde{G}$. Let S denote the unique maximal torus in G containing $\mathcal{N}(\widetilde{s})$. Then*

$$(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_\varepsilon(\widetilde{s}) = \sum_{w\in W_k(G,T)\backslash W_k(\widetilde{G},S,T)} (\mathrm{R}_T^G\,\theta)(w\mathcal{N}(\widetilde{s})w^{-1}).$$

Here $W_k(\widetilde{G},S,T)$ is defined to be the quotient $\widetilde{T}(k)\backslash\{\widetilde{g} \in \widetilde{G}(k) \mid \widetilde{g}S\widetilde{g}^{-1} = T\}$ and $W_k(G,T) = N_{G(k)}(T)/T(k)$. We see that $W_k(\widetilde{G},S,T)$ is a union of right $W_k(G,T)$-cosets from the fact that $N_{G(k)}(T)\cap\widetilde{T}(k) = T(k)$, so the index set for the summation makes sense. We define $w\mathcal{N}(\widetilde{s})w^{-1}$ to be $n\mathcal{N}(\widetilde{s})n^{-1}$ for any lift n of w to $\widetilde{G}(k)$, and claim that the right-hand side does not depend on our choices of lifts.

Here is what we mean by $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_\varepsilon$. Since $\widetilde{T} \subseteq \widetilde{G}$ is an ε-invariant maximal k-torus, and $\widetilde{\theta}$ is an ε-invariant character of $\widetilde{T}(k)$, we have that ε preserves $(\widetilde{T},\widetilde{\theta})$, so it acts on the corresponding Deligne-Lusztig variety, and thus on the virtual representation whose character is $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta}$. That is, even if this representation is reducible, we can form its ε-twisted character. One can construct this character as follows. Extend $\widetilde{\theta}$ to a character of $\widetilde{T}(k)\rtimes\Gamma$ by setting $\widetilde{\theta}(\varepsilon) = 1$. Define the ε-twisted Deligne-Lusztig character $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_\varepsilon$ induced from $\widetilde{\theta}$ by $(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_\varepsilon(g) = (\mathrm{R}_{\widetilde{T}\rtimes\Gamma}^{\widetilde{G}\rtimes\Gamma}\,\widetilde{\theta})(g\varepsilon)$ for $g \in \widetilde{G}(k)$. (See [2] for the definition of Deligne-Lusztig induction for nonconnected groups.)

Remark 2. Consider the automorphism of GL(2) given by $\varepsilon(g) = {}^t g^{-1}$. Then an analogous relation holds for all irreducible representations, not just those of Deligne-Lusztig type. On the other hand, while the relation can hold for unipotent elements, it fails if $\widetilde{s}$ is regular but $\mathcal{N}(\widetilde{s})$ is singular.

Remark 3. Consider the special case where E/k is a finite extension, and $\widetilde{G}$ arises from G via restriction of scalars: $\widetilde{G} = R_{E/k}G$. Suppose that ε is the k-automorphism of $\widetilde{G}$ associated to the action of a generator of the Galois group $\mathrm{Gal}(E/k)$. Given a representation π of $G(k)$, one often has an associated representation $\widetilde{\pi}$ of $\widetilde{G}(k)$, known as the *Shintani lift* of π. (See [4] for a discussion.) The character of π and the ε-twisted character of $\widetilde{\pi}$ are related by the Shintani relation: $\Theta_\pi(\mathcal{N}(g)) = \Theta_{\widetilde{\pi},\varepsilon}(g)$. From work of Digne [3, Cor. 3.6], one already knows that if $\mathrm{R}_T^G\,\theta$ has a Shintani lift,

then it must be $\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta}$. Thus, our character relation is a generalization of Shintani's, if one ignores the fact that the former only holds at sufficiently regular elements of $\widetilde{G}(k)$, while that latter holds at all elements, provided that one interprets the norm map $\mathcal{N}$ correctly.

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1. Consequences of regularity

From now on, we let k, $\widetilde{G}$, ε, ℓ, G, and $\mathcal{N}$ be as in §0. Let Γ be the group generated by ε.

Proposition 4. *Suppose $\widetilde{s}$ belongs to an ε-invariant maximal k-torus $\widetilde{S}$ in $\widetilde{G}$, and $s := \mathcal{N}(\widetilde{s})$ is regular in $\widetilde{G}$. Let $S := (\widetilde{S} \cap G)$. Then:*

(a) $\widetilde{S}$ belongs to an ε-invariant Borel-torus pair for $\widetilde{G}$;
(b) $\widetilde{s}\varepsilon$ is regular in $\widetilde{G} \rtimes \Gamma$; and
(c) $N_{\widetilde{G}(k)}(S) = \{g \in \widetilde{G}(k) \,|\, gsg^{-1} \in \mathrm{Im}(\mathcal{N}|_{\widetilde{S}(k)})\}$.

Proof. Since $\widetilde{S}$ is ε-invariant, we have that $s \in S$. Let S' be a maximal k-torus in G containing S, and let $\widetilde{S}' = C_{\widetilde{G}}(S')$. From Proposition 1, $\widetilde{S}'$ is a maximal torus in $\widetilde{G}$, and belongs to an ε-invariant Borel-torus pair. But

$$\widetilde{S}' = C_{\widetilde{G}}(S') \subseteq C_{\widetilde{G}}(S) \subseteq C_{\widetilde{G}}(s) = \widetilde{S},$$

proving statement (a).

Now choose an ε-invariant Borel subgroup of $\widetilde{G}$ containing $\widetilde{S}$, and let Φ^+ denote the corresponding system of positive roots. For each $\alpha \in \Phi^+$, consider the following sum of root spaces in the Lie algebra $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$: $V_\alpha := \bigoplus_{\beta\in\Gamma\cdot\alpha} \widetilde{\mathfrak{g}}_\beta$. Via the adjoint action, the element $\widetilde{s}\varepsilon$ induces a linear transformation on $\widetilde{\mathfrak{g}}$, preserving each V_α. To see that $\widetilde{s}\varepsilon$ is regular, it is enough to choose a positive root $\alpha \in \Phi^+$ and show that the action of $\widetilde{s}\varepsilon$ on V_α does not have 1 as an eigenvalue.

Let $m = m_\alpha = |\Gamma\cdot\alpha|$. There is a nonzero scalar $\mu = \mu_\alpha$, not necessarily rational, such that ε^m acts on V_α via multiplication by μ. By necessity, μ is a root of unity of order dividing ℓ/m. Let $\chi' = \chi'_\alpha = \sum_{\beta\in\Gamma\cdot\alpha}\beta$, and let $\chi = \chi_\alpha = \sum_{\gamma\in\Gamma}\gamma(\alpha)$. Thus, $\chi = \frac{\ell}{m}\chi'$.

It is not hard to see that the characteristic polynomial (in the variable X) of our transformation of V_α is $X^m - \mu\cdot\chi'(\widetilde{s})$, so it is enough to show that $\mu\cdot\chi'(\widetilde{s}) \neq 1$.

Since s is regular, we have that

$$1 \neq \alpha(s) = \chi(\widetilde{s}) = \chi'(\widetilde{s})^{\ell/m} = (\mu\cdot\chi'(\widetilde{s}))^{\ell/m},$$

and therefore, $\mu\cdot\chi'(\widetilde{s}) \neq 1$, proving statement (b).

Now consider statement (c). From statement (a) and Proposition 1, we have that S is a maximal k-torus in G, and $\widetilde{S} = C_{\widetilde{G}}(S)$. To show that the right-hand-side is included in the left, take any $g \in \widetilde{G}$ such that $gsg^{-1} \in \mathrm{Im}(\mathcal{N}|_{\widetilde{S}(k)}) \subseteq S(k)$. Since s

and gsg^{-1} are in $S(k)$, they are both ε-invariant, so we have $\varepsilon(g)s\varepsilon(g)^{-1} = gsg^{-1}$. This implies $g^{-1}\varepsilon(g) \in C_{\widetilde{G}}(s) = \widetilde{S}$. Therefore $\varepsilon(g) = g\widetilde{t}$ for some $\widetilde{t} \in \widetilde{S}(k)$.

Since g conjugates a regular element of $\widetilde{S}$ back into $\widetilde{S}$, we must have that g normalizes $\widetilde{S}$. Let $t \in S(k)$. Since

$$\varepsilon(gtg^{-1}) = g\widetilde{t}t\widetilde{t}^{-1}g^{-1} = gtg^{-1},$$

we have that $gtg^{-1} \in \widetilde{S}^{\varepsilon}$. But since $t \in (\widetilde{S}^{\varepsilon})^{\circ} = S$, so is gtg^{-1}. Thus, $g \in N_{\widetilde{G}}(S)$.

To prove the converse, let $n \in N_{\widetilde{G}(k)}(S)$. We want to show that $nsn^{-1} \in \operatorname{Im}(\mathcal{N}|_{\widetilde{S}(k)})$. Since every element in S is ε-fixed, we have $\varepsilon^i(n)s\varepsilon^i(n)^{-1} = nsn^{-1}$ for all i, and therefore $n^{-1}\varepsilon^i(n)$ belongs to $C_{\widetilde{G}(k)}(s) = \widetilde{S}$, and thus commutes with every element of $\widetilde{S}(k)$. Consider the following product:

$$\left(n^{-1}\varepsilon(n)\right)\left(\varepsilon(n)^{-1}\varepsilon^2(n)\right)\left(\varepsilon^2(n)^{-1}\varepsilon^3(n)\right)\cdots\left(\varepsilon^{\ell-1}(n)^{-1}n\right) = 1$$

Since each term in the above equation commutes with $\varepsilon^j(\widetilde{s})$, we have

$$\begin{aligned}
nsn^{-1} &= n\widetilde{s}\varepsilon(\widetilde{s})\cdots\varepsilon^{\ell-1}(\widetilde{s})n^{-1}\\
&= n\widetilde{s}\left(n^{-1}\varepsilon(n)\right)\varepsilon(\widetilde{s})\left(\varepsilon(n)^{-1}\varepsilon^2(n)\right)\cdots\varepsilon^{\ell-1}(\widetilde{s})\left(\varepsilon^{\ell-1}(n)^{-1}n\right)n^{-1}\\
&= \left(n\widetilde{s}n^{-1}\right)\left(\varepsilon(n)\varepsilon(\widetilde{s})\varepsilon(n)^{-1}\right)\cdots\left(\varepsilon^{\ell-1}(n)\varepsilon^{\ell-1}(\widetilde{s})\varepsilon^{l-1}(n)^{-1}\right)\\
&= \left(n\widetilde{s}n^{-1}\right)\left(\varepsilon(n\widetilde{s}n^{-1})\right)\cdots\left(\varepsilon^{\ell-1}(n\widetilde{s}n^{-1})\right)\\
&= \mathcal{N}(n\widetilde{s}n^{-1}).
\end{aligned}$$

Since n conjugates a regular element (s) of $\widetilde{S}$ back into $\widetilde{S}$, we must have that $n \in N_{\widetilde{G}}(\widetilde{S})$, and so $n\widetilde{s}n^{-1} \in \widetilde{S}(k)$. Therefore

$$nsn^{-1} = \mathcal{N}(n\widetilde{s}n^{-1}) \in \operatorname{Im}(\mathcal{N}|_{\widetilde{S}(k)}),$$

as desired. $\square$

2. Proof of the Theorem

We start by recalling some facts about Deligne-Lusztig virtual characters.

Lemma 5.

(a) If $x \in G(k)$ is a regular semisimple element, then

$$(\mathrm{R}_T^G\,\theta)(x) = \begin{cases} \displaystyle\sum_{v \in W_k(G,T)} \theta(vx'v^{-1}) & \text{if } x \text{ is } G(k)\text{-conjugate to an element } x' \in T(k),\\ 0 & \text{otherwise.}\end{cases}$$

(b) Suppose that $\widetilde{x} \in \widetilde{G}(k)$ and that $\widetilde{x}\varepsilon$ is a regular semisimple element of $\widetilde{G}(k) \rtimes \Gamma$. Let $\widetilde{\theta}$ be an ε-invariant character of $\widetilde{T}(k)$, extended to trivially $\widetilde{T}(k) \rtimes \Gamma$. Then

$$(\mathrm{R}_{\widetilde{T}}^{\widetilde{G}}\,\widetilde{\theta})_{\varepsilon}(\widetilde{x}) = \frac{1}{|\widetilde{T}(k) \rtimes \Gamma|}\sum_{\left\{h \in \widetilde{G}(k)\rtimes\Gamma \,\middle|\, \widetilde{x}\varepsilon \in h(\widetilde{T}(k)\rtimes\Gamma)h^{-1}\right\}} \widetilde{\theta}(h^{-1}(\widetilde{x}\varepsilon)h).$$

Proof. For each formula, see [2, Proposition 2.6]. For the second formula, note that $C_{\widetilde{G}}(x)$ contains no nontrivial unipotent elements and that the Green function $Q^{C_{\widetilde{G}}(\widetilde{x}\varepsilon)^{\circ}(k)}_{C_{\widetilde{G}}(\widetilde{x}\varepsilon)^{\circ}(k)}$ in this proposition takes the value $|C_{\widetilde{G}}(\widetilde{x}\varepsilon)^{\circ}(k)|$ at $(1,1)$. $\square$

Let $\widetilde{S}$ denote an ε-invariant maximal k-torus in $\widetilde{G}$ containing $\widetilde{s}$. From Proposition 4(a) and Proposition 1, $S = \widetilde{S} \cap G$ and $\widetilde{S} = C_{\widetilde{G}}(S)$.

Suppose that for all $\widetilde{h} \in \widetilde{G}(k) \rtimes \Gamma$, we have that $\widetilde{h}(\widetilde{s}\varepsilon)\widetilde{h}^{-1} \notin \widetilde{T}(k)\varepsilon$. Then $\widetilde{h}\widetilde{s}\varepsilon(\widetilde{h}^{-1}) \notin \widetilde{T}(k)$, so in particular $g\widetilde{s}g^{-1} \notin \widetilde{T}(k)$ for all $g \in G(k)$. This implies that for all $g \in G(k)$, $g\widetilde{S}g^{-1} \neq \widetilde{T}$, and thus from Proposition 1 that $gSg^{-1} \neq T$. Then the regularity of $\mathcal{N}(\widetilde{s})$ implies that $g\mathcal{N}(\widetilde{s})g^{-1} \notin T(k)$. For every lift n of w to $\widetilde{G}(k)$, we have that the summand corresponding to w in the right-hand side of the equation in the theorem is zero. But our assumption on $\widetilde{s}$ implies that the left-hand side is zero, too. Thus, the theorem holds in this case.

Now suppose that $\widetilde{h}(\widetilde{s}\varepsilon)\widetilde{h}^{-1} \in \widetilde{T}(k)\varepsilon$ for some $\widetilde{h} = \widetilde{g}\varepsilon^i \in \widetilde{G}(k) \rtimes \Gamma$. Then $\widetilde{g} \cdot \widetilde{s} \cdot \varepsilon(\widetilde{g})^{-1} \in \widetilde{T}(k)$ so $\widetilde{g}\mathcal{N}(\widetilde{s})\widetilde{g}^{-1} = \mathcal{N}(\widetilde{g} \cdot \widetilde{s} \cdot \varepsilon(\widetilde{g})^{-1}) \in T(k)$. Thus $\widetilde{g}S\widetilde{g}^{-1} = T$. Rewriting each index w in the summation in the theorem in the form $w'\widetilde{g}$, where w' is a coset representative for $W_k(G,T)\backslash W_k(\widetilde{G},T)$ (where $W_k(\widetilde{G},T)$ is defined to be $W_k(\widetilde{G},T,T)$), we see that the right-hand side is equal to

$$\sum_{w \in W_k(G,T)\backslash W_k(\widetilde{G},T)} (\mathrm{R}_T^G\,\theta)(w\widetilde{g}\mathcal{N}(\widetilde{s})\widetilde{g}^{-1}w^{-1}).$$

Thus, we may replace $\widetilde{s}$ by its twisted conjugate $\widetilde{h} \cdot \widetilde{s} \cdot \varepsilon(\widetilde{h}^{-1})$, and $\mathcal{N}(\widetilde{s})$ by its conjugate $\widetilde{g}\mathcal{N}(\widetilde{s})\widetilde{g}^{-1}$. Having done so, we may now assume that $S = T$ and $\widetilde{S} = \widetilde{T}$.

From Proposition 4(b), we have that $\widetilde{s}\varepsilon$ is regular, so we can analyze the right-hand side of the equation (LHS) using Lemma 5(b):

$$\mathrm{LHS} = \frac{1}{\ell|\widetilde{T}(k)|}\sum \widetilde{\theta}(h^{-1}(\widetilde{s}\varepsilon)h) = \frac{1}{\ell|\widetilde{T}(k)|}\sum \widetilde{\theta}\left(h^{-1}\widetilde{s}\varepsilon(h)\varepsilon\right),$$

where each sum is over the set $\left\{h \in \widetilde{G}(k) \rtimes \Gamma \,\middle|\, \widetilde{s}\varepsilon \in h\big(\widetilde{T} \rtimes \Gamma\big)h^{-1}\right\}$. If $g \in \widetilde{G}(k)$, then $g\varepsilon^i$ belongs to the index set if and only if $g^{-1}\widetilde{s}\varepsilon(g) \in \widetilde{T}(k)$. Thus,

$$\begin{aligned}
\mathrm{LHS} &= \frac{1}{\ell|\widetilde{T}(k)|}\sum_{i=0}^{\ell-1}\sum_{\left\{g \in \widetilde{G}(k) \,\middle|\, g^{-1}\widetilde{s}\varepsilon(g) \in \widetilde{T}(k)\right\}} \widetilde{\theta}\left((g\varepsilon^i)^{-1}\widetilde{s}\varepsilon(g\varepsilon^i)\varepsilon\right)\\
&= \frac{1}{|\widetilde{T}(k)|}\sum_{\left\{g \in \widetilde{G}(k) \,\middle|\, g^{-1}\widetilde{s}\varepsilon(g) \in \widetilde{T}(k)\right\}} \widetilde{\theta}(g^{-1}\widetilde{s}\varepsilon(g))
\end{aligned}$$

Letting $s = \mathcal{N}(\widetilde{s})$, we hence have

$$\begin{aligned}
\mathrm{LHS} &= \frac{1}{|\widetilde{T}(k)|}\sum_{\widetilde{t} \in \widetilde{T}(k)}\sum_{\left\{g \in \widetilde{G}(k) \,\middle|\, g^{-1}\widetilde{s}\varepsilon(g) = \widetilde{t}\right\}} \widetilde{\theta}(g^{-1}\widetilde{s}\varepsilon(g))\\
&= \frac{1}{|\widetilde{T}(k)|}\sum_{\widetilde{t} \in \widetilde{T}(k)}\sum_{\left\{g \in \widetilde{G}(k) \,\middle|\, g\widetilde{s}\varepsilon(g)^{-1} = \widetilde{t}\right\}} \widetilde{\theta}(g\widetilde{s}\varepsilon(g)^{-1})\\
&= \frac{1}{|\widetilde{T}(k)|}\sum_{\widetilde{t} \in \widetilde{T}(k)}\sum_{\left\{g \in \widetilde{G}(k) \,\middle|\, g\widetilde{s}\varepsilon(g)^{-1} = \widetilde{t}\right\}} \theta(\mathcal{N}(\widetilde{t}))\\
&= \frac{1}{|\widetilde{T}(k)|}\sum_{\widetilde{t} \in \widetilde{T}(k)} \left|\left\{g \in \widetilde{G}(k) \,\middle|\, g\widetilde{s}\varepsilon(g)^{-1} = \widetilde{t}\right\}\right| \theta(\mathcal{N}(\widetilde{t}))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\widetilde{T}(k)|} \sum_{t \in T(k)} \left| \left\{ g \in \widetilde{G}(k) \,\middle|\, g\widetilde{s}\varepsilon(g)^{-1} \in \widetilde{T} \text{ and } gsg^{-1} = t \right\} \right| \theta(t) \\
&= \frac{1}{|\widetilde{T}(k)|} \sum_{t \in \mathrm{Im}(\mathcal{N}|_{\widetilde{T}(k)})} \left| \left\{ g \in \widetilde{G}(k) \,\middle|\, gsg^{-1} = t \right\} \right| \theta(t) \\
&= \frac{1}{|\widetilde{T}(k)|} \sum_{\left\{ g \in \widetilde{G}(k) \,\middle|\, gsg^{-1} \in \mathrm{Im}(\mathcal{N}|_{\widetilde{T}(k)}) \right\}} \theta(gsg^{-1}) \\
&= \frac{1}{|\widetilde{T}(k)|} \sum_{g \in N_{\widetilde{G}(k)}(T)} \theta(gsg^{-1}) \qquad \text{(by Proposition 4(c))} \\
&= \sum_{w \in W_k(\widetilde{G},T)} \theta(wsw^{-1}) \\
&= \sum_{w \in W_k(G,T)\backslash W_k(\widetilde{G},T)} \sum_{v \in W_k(G,T)} \theta(vwsw^{-1}v^{-1}),
\end{aligned}
$$

where the first summation in the final line is over a set of representatives for the right-coset space $W_k(G,T)\backslash W_k(\widetilde{G},T)$. But from Lemma 5(a), the final expression above is equal to the right-hand side of the relation in the Theorem. □

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